# IAL C34 JUNE 15

**1.** A curve has equation

$$4x^2 - y^2 + 2xy + 5 = 0$$

The points $P$ and $Q$ lie on the curve.

Given that $\dfrac{dy}{dx} = 2$ at $P$ and at $Q$,

(a) use implicit differentiation to show that $y - 6x = 0$ at $P$ and at $Q$.

**(6)**

(b) Hence find the coordinates of $P$ and $Q$.

**(3)**

a) $$8x - 2y\frac{dy}{dx} + 2x\frac{dy}{dx} + 2y = 0$$

$$\frac{dy}{dx}(2x - 2y) = -2y - 8x$$

$$\frac{dy}{dx} = \frac{-y - 4x}{x - y} = 2$$

$$-y - 4x = 2x - 2y$$

$$\boxed{y - 6x = 0}$$

b) $$y = 6x$$

$$4x^2 - 36x^2 + 2x(6x) + 5 = 0$$

$$20x^2 = 5$$

$$x^2 = \frac{1}{4} \rightarrow x = \pm\frac{1}{2}$$

$$\therefore\ P\left(-\frac{1}{2}, -3\right)$$

$$Q\left(\frac{1}{2}, 3\right)$$

2. Given that

$$\frac{4(x^2+6)}{(1-2x)(2+x)^2} \equiv \frac{A}{(1-2x)} + \frac{B}{(2+x)} + \frac{C}{(2+x)^2}$$

(a) find the values of the constants $A$ and $C$ and show that $B = 0$

**(4)**

(b) Hence, or otherwise, find the series expansion of

$$\frac{4(x^2+6)}{(1-2x)(2+x)^2}, \quad |x| < \frac{1}{2}$$

in ascending powers of $x$, up to and including the term in $x^2$, simplifying each term.

**(5)**

a) $4(x^2+6) = A(2+x)^2 + B(1-2x)(2+x) + C(1-2x)$

let $x = \frac{1}{2}$

$25 = \frac{25}{4}A \longrightarrow \boxed{A=4}$

let $x = -2$

$40 = 5C \longrightarrow \boxed{C=8}$

Coefficient of $x^2$: $4 = A - 2B$

$4 = 4 - 2B \longrightarrow \boxed{B=0}$

b) $4(1-2x)^{-1} = 4\left[1 + 2x + \frac{(-1)(-2)}{2!}(-2x)^2\right]$

$= 4 + 8x + 16x^2$

$8(2+x)^{-2} = 8(2^{-2})\left(1+\frac{x}{2}\right)^{-2}$

$= 2\left[1 - x + \frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2\right]$

$= 2 - 2x + \frac{3}{2}x^2$

$4(1-2x)^{-1} + 8(2+x)^{-2} = 6 + 6x + \frac{35}{2}x^2$

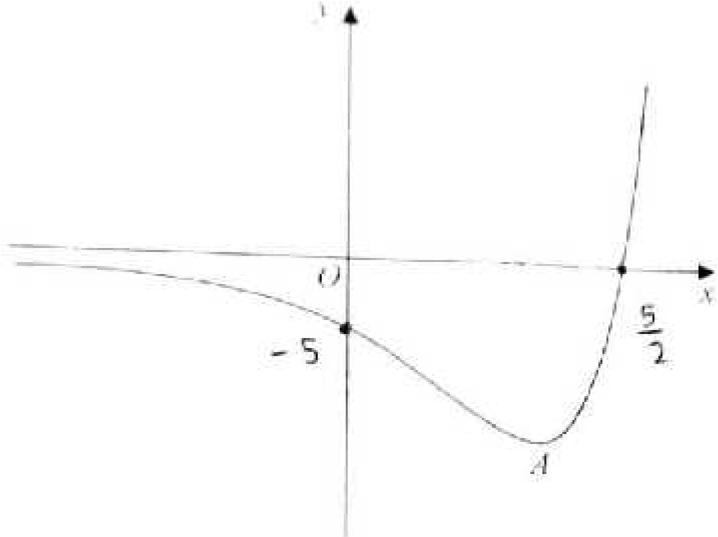



**Figure 1**

Figure 1 shows a sketch of part of the curve with equation $y = f(x)$, where

$$f(x) = (2x - 5)e^x, \quad x \in \mathbb{R}$$

The curve has a minimum turning point at $A$.

(a) Use calculus to find the exact coordinates of $A$.

**(5)**

Given that the equation $f(x) = k$, where $k$ is a constant, has exactly two roots,

(b) state the range of possible values of $k$.

**(2)**

(c) Sketch the curve with equation $y = |f(x)|$.

Indicate clearly on your sketch the coordinates of the points at which the curve crosses or meets the axes.

**(3)**

a) $f'(x) = 2e^x + (2x-5)e^x = 0$

$e^x(-3+2x) = 0 \rightarrow x = \frac{3}{2}$

$y = (2(\frac{3}{2}) - 5)e^{3/2}$

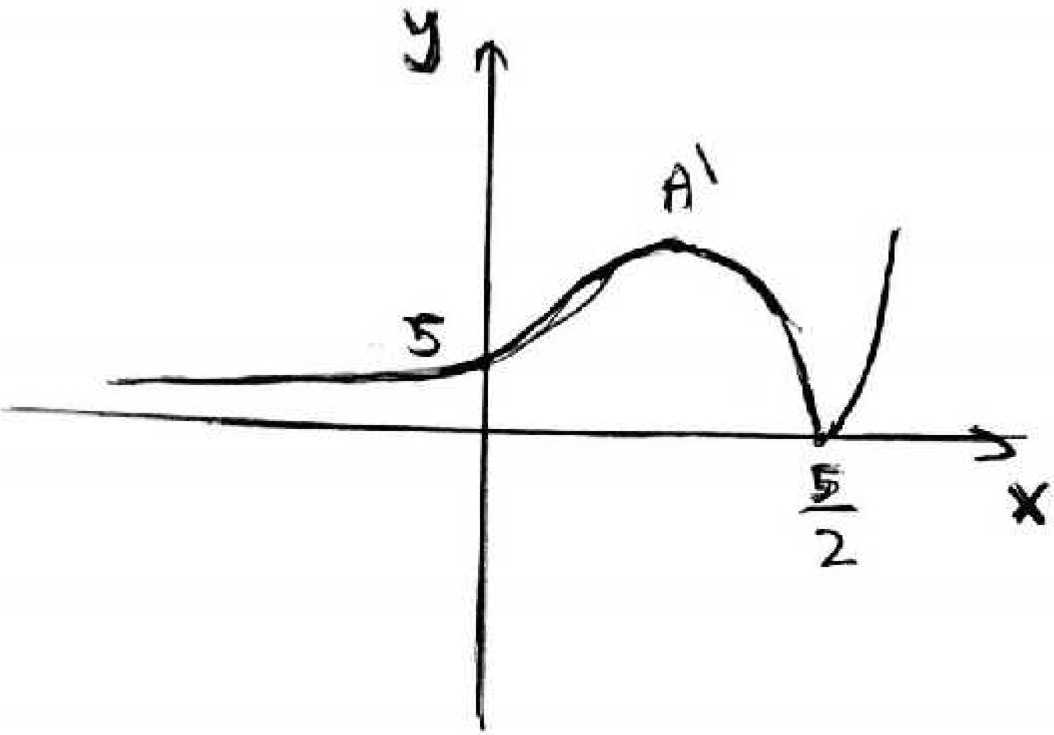
$= -2e^{3/2}$

$A(\frac{3}{2}, -2e^{3/2})$

b) $y = 0$ is a asymptote

$\therefore -2e^{3/2} < k < 0$

Question 3 continued



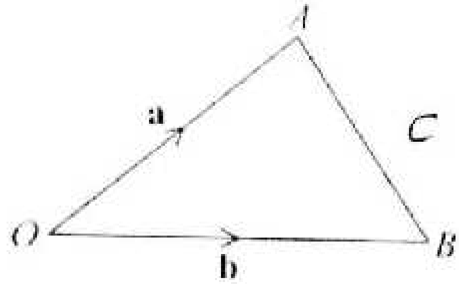



**Figure 2**

Figure 2 shows the points $A$ and $B$ with position vectors **a** and **b** respectively, relative to a fixed origin $O$.

Given that $|\mathbf{a}| = 5$, $|\mathbf{b}| = 6$ and $\mathbf{a}.\mathbf{b} = 20$

(a) find the cosine of angle $AOB$.

**(2)**

(b) find the exact length of $AB$.

**(2)**

(c) Show that the area of triangle $OAB$ is $5\sqrt{5}$

**(3)**

a) $\cos AOB = \dfrac{a \cdot b}{|a|\,|b|} = \dfrac{20}{30} = \dfrac{2}{3}$

b) $c^2 = a^2 + b^2 - 2ab\cos AOB$

$= 25 + 36 - 60\left(\frac{2}{3}\right) = 21$

$AB = \sqrt{21}$

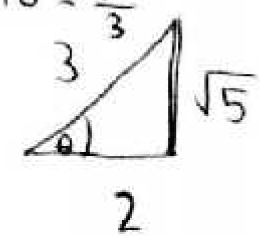
c) $\text{Area} = \frac{1}{2}ab\sin AOB$

$\cos\theta = \frac{2}{3}$

$\sin\theta = \frac{\sqrt{5}}{3}$

$= \frac{1}{2}(5)(6)\left(\frac{\sqrt{5}}{3}\right)$

$= 15\left(\frac{\sqrt{5}}{3}\right) = \boxed{5\sqrt{5}}$



5. (i) Find the $x$ coordinate of each point on the curve $y = \frac{x}{x+1}$, $x \neq -1$, at which the gradient is $\frac{1}{4}$

**(4)**

(ii) Given that

$$\int_a^{2a} \frac{t+1}{t} \, dt = \ln 7 \qquad a > 0$$

find the exact value of the constant $a$.

**(4)**

i) $y' = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} = \frac{1}{4}$

$4 = (x+1)^2$

$\pm 2 = x + 1$

$x = 1$ or $x = -3$

ii) $\int_a^{2a} \left(1 + \frac{1}{t}\right) dt = \left[t + \ln|t|\right]_a^{2a}$

$= 2a + \ln(2a) - a - \ln(a)$

$= a + \ln(2) + \cancel{\ln(a)} - \cancel{\ln(a)}$

$a + \ln 2 = \ln 7$

$\boxed{a = \ln\left(\frac{7}{2}\right)}$

6. The mass, $m$ grams, of a radioactive substance $t$ years after first being observed, is modelled by the equation

$$m = 25e^{1-kt}$$

where $k$ is a positive constant.

(a) State the value of $m$ when the radioactive substance was first observed.

**(1)**

Given that the mass is 50 grams, 10 years after first being observed,

(b) show that $k = \frac{1}{10}\ln\left(\frac{1}{2}e\right)$

**(4)**

(c) Find the value of $t$ when $m = 20$, giving your answer to the nearest year.

**(3)**

a) $t=0 \rightarrow m = 25e$

b) $50 = 25e^{1-10k}$

$e^{1-10k} = 2$

$1-10k = \ln(2)$

$-10k = \ln(2) - 1$

$k = \frac{1}{10} - \frac{1}{10}\ln(2)$

$= \frac{1}{10}\ln e - \frac{1}{10}\ln(2)$

$= \frac{1}{10}\left[\ln e - \ln 2\right] = \frac{1}{10}\ln\frac{e}{2}$

c) $\frac{20}{25} = e^{1-kt}$

$\ln\left(\frac{4}{5}\right) = 1-kt$

$kt = 1 - \ln\left(\frac{4}{5}\right) \Rightarrow t = \frac{1}{k}\left[1 - \ln\left(\frac{4}{5}\right)\right]$

$t = 39.86\ldots$

Leave blank

7. (a) Use the substitution $t = \tan x$ to show that the equation

$$4\tan 2x - 3\cot x \sec^2 x = 0$$

can be written in the form

$$3t^4 + 8t^2 - 3 = 0$$

**(4)**

(b) Hence solve, for $0 \leqslant x < 2\pi$,

$$4\tan 2x - 3\cot x \sec^2 x = 0$$

Give each answer in terms of $\pi$. You must make your method clear.

**(4)**

a) $4\left(\frac{2\tan x}{1-\tan^2 x}\right) - 3\cot x\left(1+\tan^2 x\right) = 0$

$= \frac{8\tan x}{1-\tan^2 x} - 3\frac{1}{\tan x} - 3\tan x = 0$

let $t = \tan x$

$\frac{8t}{1-t^2} - \frac{3}{t} - 3t = 0 \quad * \, t(1-t^2)$

$8t^2 - 3(1-t^2) - 3t^2(1-t^2) = 0$

$8t^2 - 3 + \cancel{3t^2} - \cancel{3t^2} + 3t^4 = 0$

$3t^4 + 8t^2 - 3 = 0$

b)

$(3t^2 - 1)(t^2 + 3) = 0$

$t^2 = \frac{1}{3} \qquad t^2 = -3$

$\tan x = \frac{1}{\sqrt{3}} \quad , \tan x = -\frac{1}{\sqrt{3}} \quad$ rejected

$x = \boxed{\frac{1}{6}\pi} \qquad x = -\frac{1}{6}\pi \qquad -\frac{1}{6}\pi + 2\pi = \boxed{\frac{11}{6}\pi}$

$\pi + \frac{1}{6}\pi = \boxed{\frac{7}{6}\pi} \qquad \pi + \left(-\frac{1}{6}\pi\right) = \boxed{\frac{5}{6}\pi}$

$x = \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{7}{6}\pi, \frac{11}{6}\pi$

Leave blank

8. (a) Prove by differentiation that

$$\frac{d}{dy}(\ln \tan 2y) = \frac{4}{\sin 4y}, \qquad 0 < y < \frac{\pi}{4}$$

**(4)**

(b) Given that $y = \frac{\pi}{6}$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = 2\cos x \sin 4y, \qquad 0 < y < \frac{\pi}{4}$$

Give your answer in the form $\tan 2y = Ae^{B\sin x}$, where $A$ and $B$ are constants to be determined.

**(6)**

a) $\frac{d}{dy}(\ln \tan 2y) = \frac{2\sec^2 2y}{\tan 2y} = \frac{\frac{2}{\cos^2 2y}}{\frac{\sin 2y}{\cos 2y}}$

$= \frac{2}{\sin 2y \cos 2y} = \frac{4}{2\sin 2y \cos 2y} = \boxed{\frac{4}{\sin 4y}}$

b) $\int \frac{1}{\sin 4y}\, dy = \int 2\cos x\, dx$

$\frac{\ln(\tan 2y)}{4} = 2\sin x + C$

$x = 0,\ y = \frac{\pi}{6} \rightarrow \qquad \frac{\ln\sqrt{3}}{4} = 0 + C$

$C = \frac{1}{8}\ln(3)$

$\ln \tan 2y = 8\sin x + \frac{1}{2}\ln(3)$

$\tan 2y = e^{8\sin x + \ln\sqrt{3}}$

$= e^{8\sin x}\, e^{\ln\sqrt{3}}$

$\boxed{\tan 2y = \sqrt{3}\, e^{8\sin x}}$

9.

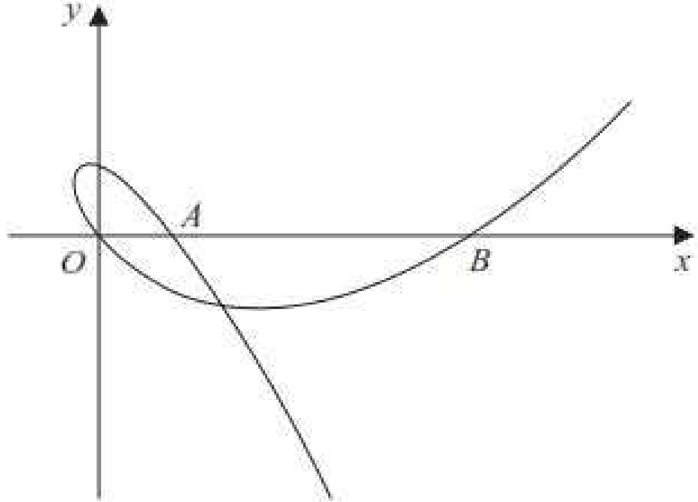



**Figure 3**

Figure 3 shows a sketch of part of the curve with parametric equations

$$x = t^2 + 2t, \qquad y = t^3 - 9t, \qquad t \in \mathbb{R}$$

The curve cuts the $x$-axis at the origin and at the points $A$ and $B$ as shown in Figure 3.

(a) Find the coordinates of point $A$ and show that point $B$ has coordinates $(15, 0)$.

**(3)**

(b) Show that the equation of the tangent to the curve at $B$ is $9x - 4y - 135 = 0$

**(5)**

The tangent to the curve at $B$ cuts the curve again at the point $X$.

(c) Find the coordinates of $X$.

**(5)**

(*Solutions based entirely on graphical or numerical methods are not acceptable.*)

a) $0 = t^3 - 9t$ , $t(t^2 - 9) = 0$

$t = 0$, $t = \pm 3$

$t = -3 \to x = 9 - 6 = 3$ $\qquad$ $t = 3 \to x = 15$

$A(3, 0)$ $\qquad$ $B(15, 0)$

b) $\frac{dx}{dt} = 2t + 2$ $\qquad$ $\frac{dy}{dt} = 3t^2 - 9$

$\frac{dy}{dt} = \frac{3t^2 - 9}{2t + 2}$ $\qquad$ at $t = 3 \to y' = \frac{9}{4}$

$\frac{y - 0}{x - 15} = \frac{9}{4}$

c) $9(t^2+2t)-4(t^3-9t)-135=0$

$$9t^2+18t-4t^3+36t-135=0$$
$$-4t^3+9t^2+54t-135=0$$
$$4t^3-9t^2-54t+135=0$$

$$f(t)=4t^3-9t^2-54t+135$$
$$f(3)=4(3)^3-9(3)^2-54(3)+135$$
$$=108-81-162+135$$
$$=27-162+135=0$$

$\therefore$ $t-3$ is a factor

$$\begin{array}{r l}
 & 4t^2+3t-45 \\
t-3 & \overline{)\,4t^3-9t^2-54t+135} \\
\ominus & 4t^3-12t^2 \\ \hline
 & 3t^2-54t \\
\ominus & 3t^2-9t \\ \hline
 & -45t+135 \\
\ominus & -45t+135 \\ \hline
 & 0 \quad 0
\end{array}$$

$$(t-3)(4t^2+3t-45)=0$$
$$(t-3)(4t+15)(t-3)=0$$
$$(4t+15)(t-3)^2=0$$

$$-4t=15$$
$$t=-\frac{15}{4}$$

$$x=\left(-\frac{15}{4}\right)^2+2\left(-\frac{15}{4}\right)=\frac{105}{16}$$

$$y=\left(-\frac{15}{4}\right)^3-9\left(-\frac{15}{4}\right)=-\frac{1215}{64}$$

$$X\left(\frac{105}{16},-\frac{1215}{64}\right)$$

10.

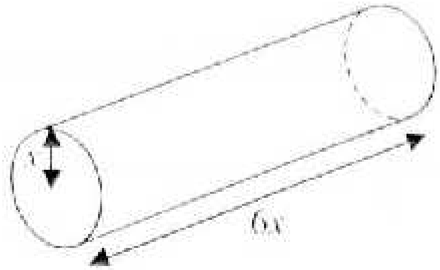



**Figure 4**

Figure 4 shows a right circular cylindrical rod which is expanding as it is heated.

At time $t$ seconds the radius of the rod is $x$ cm and the length of the rod is $6x$ cm.

Given that the **cross-sectional area** of the rod is increasing at a constant rate of $\frac{\pi}{20}$ cm$^2$ s$^{-1}$, find the rate of increase of the volume of the rod when $x = 2$

Write your answer in the form $k\pi$ cm$^3$ s$^{-1}$ where $k$ is a rational number.

**(6)**

$$\frac{dA}{dt} = \frac{\pi}{20}$$

$$\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$$

$$= \frac{9}{\sqrt{\pi}} \sqrt{A} \left(\frac{\pi}{20}\right)$$

$$= \frac{9}{20} \sqrt{A} \sqrt{\pi}$$

$x=2$

$$= \frac{9}{20} \sqrt{4\pi} \sqrt{\pi}$$

$$\frac{dV}{dt} = \boxed{\frac{9}{10}\pi}$$

$$V = 6x(\pi x^2)$$

$$A = \pi x^2$$

$$x = \sqrt{\frac{A}{\pi}}$$

$$V = 6x(A)$$

$$V = 6\left(\sqrt{\frac{A}{\pi}}\right)(A)$$

$$V = \frac{6}{\sqrt{\pi}} A^{3/2}$$

$$\frac{dV}{dA} = \frac{9}{\sqrt{\pi}} A^{1/2}$$

$$x = 2 \rightarrow A = 4\pi$$

**11.** (a) Express $1.5\sin\theta - 1.2\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the value of $R$ and the value of $\alpha$ to 3 decimal places.

**(3)**

The height, $H$ metres, of sea water at the entrance to a harbour on a particular day, is modelled by the equation

$$H = 3 + 1.5\sin\left(\frac{\pi t}{6}\right) - 1.2\cos\left(\frac{\pi t}{6}\right), \quad 0 \leqslant t < 12$$

where $t$ is the number of hours after midday.

(b) Using your answer to part (a), calculate the minimum value of $H$ predicted by this model and the value of $t$, to 2 decimal places, when this minimum occurs.

**(4)**

(c) Find, to the nearest minute, the times when the height of sea water at the entrance to the harbour is predicted by this model to be 4 metres.

**(6)**

a) $1.5\sin\theta - 1.2\cos\theta = R\sin\theta\cos\alpha - R\sin\alpha\cos\theta$

$R\cos\alpha = 1.5$

$R\sin\alpha = 1.2$ $\qquad R = \sqrt{1.5^2 + 1.2^2} = \frac{3\sqrt{41}}{10}$

$\tan\alpha = \frac{4}{5}$ $\qquad = 1.921$

$\alpha = 0.675$

b) $H = 3 + \frac{3\sqrt{41}}{10}\sin\left(\frac{\pi}{6}t - 0.675\right)$, $H_{min.} = 3 - \frac{3\sqrt{41}}{10}$

$\frac{\pi}{6}t - 0.675 = \frac{3}{2}\pi$ $\qquad = 1.08$ metres

$\frac{\pi}{6}t = 5.387$ $\qquad \boxed{t = 10.29 \text{ hours}}$

c) $1 = \frac{3\sqrt{41}}{10}\sin\left(\frac{\pi}{6}t - 0.675\right)$

$\frac{\pi}{6}t - 0.675 = 0.5475$, $\quad \frac{\pi}{6}t - 0.675 = \pi - 0.5475$

$t = 2.33$ h $\qquad t = 6.2435$ h

$= 139.8$ min $\qquad = 374.61$ min.

$\sim \boxed{140 \text{ min.}}$ $\qquad \sim \boxed{375 \text{ min.}}$

12. (i) Relative to a fixed origin $O$, the line $l_1$ is given by the equation

$$l_1: \mathbf{r} = \begin{pmatrix} -5 \\ 1 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \text{ where } \lambda \text{ is a scalar parameter.}$$

The point $P$ lies on $l_1$. Given that $\overrightarrow{OP}$ is perpendicular to $l_1$, calculate the coordinates of $P$.

**(5)**

(ii) Relative to a fixed origin $O$, the line $l_2$ is given by the equation

$$l_2: \mathbf{r} = \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} \text{ where } \mu \text{ is a scalar parameter.}$$

The point $A$ **does not** lie on $l_2$. Given that the vector $\overrightarrow{OA}$ is parallel to the line $l_2$ and $|\overrightarrow{OA}| = \sqrt{2}$ units, calculate the possible position vectors of the point $A$.

**(5)**

i) $\overrightarrow{OP} = \begin{pmatrix} -5+2\lambda \\ 1-3\lambda \\ 6+\lambda \end{pmatrix}$ , $\begin{pmatrix} -5+2\lambda \\ 1-3\lambda \\ 6+\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0$

$-10+4\lambda-3+9\lambda+6+\lambda=0$

$14\lambda = 7$ $\boxed{\lambda = \frac{1}{2}}$

$P = \begin{pmatrix} -4 \\ -0.5 \\ 6.5 \end{pmatrix}$

ii) $\overrightarrow{OA} = \begin{pmatrix} 5t \\ -3t \\ 4t \end{pmatrix}$ $(5t)^2 + (-3t)^2 + (4t)^2 = (\sqrt{2})^2$

$50t^2 = 2$

$t^2 = \frac{1}{25}$

$t = \pm\frac{1}{5}$

$\overrightarrow{OA} = \begin{pmatrix} -1 \\ \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ or $\begin{pmatrix} 1 \\ -\frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$

**13.**

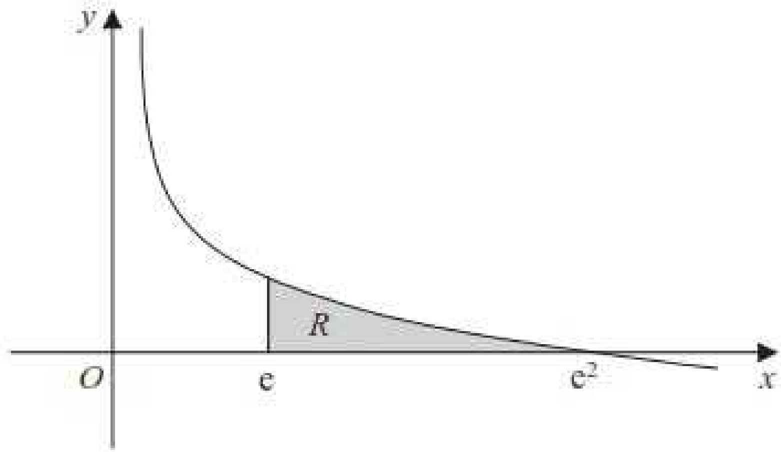



**Figure 5**

Figure 5 shows a sketch of part of the curve with equation $y = 2 - \ln x$, $x > 0$

The finite region $R$, shown shaded in Figure 5, is bounded by the curve, the $x$-axis and the line with equation $x = \mathrm{e}$.

The table below shows corresponding values of $x$ and $y$ for $y = 2 - \ln x$

| $x$ | $\mathrm{e}$ | $\frac{\mathrm{e}+\mathrm{e}^2}{2}$ | $\mathrm{e}^2$ |
|---|---|---|---|
| $y$ | 1 | | 0 |

(a) Complete the table giving the value of $y$ to 4 decimal places.

**(1)**

(b) Use the trapezium rule, with all the values of $y$ in the completed table, to obtain an estimate for the area of $R$, giving your answer to 3 decimal places.

**(3)**

(c) Use integration by parts to show that $\int (\ln x)^2 \, \mathrm{d}x = x(\ln x)^2 - 2x\ln x + 2x + c$

**(4)**

The area $R$ is rotated through 360° about the $x$-axis.

(d) Use calculus to find the exact volume of the solid generated.

Write your answer in the form $\pi\mathrm{e}(p\mathrm{e} + q)$, where $p$ and $q$ are integers to be found.

**(6)**

b) $h = \frac{e+e^2}{2} - e = \frac{e^2 - e}{2}$

$\frac{1}{2}\left(\frac{e^2-e}{2}\right)\left[1+0+2(0.3799)\right] = 2.055$

c) $I = \int (\ln x)^2 \, dx$ $\quad u = (\ln x)^2 \quad dv = dx$

$du = \frac{2\ln x}{x} dx \quad v = x$

$I = x(\ln x)^2 - \int 2\ln x \, dx$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x} dx \quad v = x$

$I = x(\ln x)^2 - 2\left[ x\ln x - \int dx \right]$

$= x(\ln x)^2 - 2x\ln x + 2x + c$

d) $V = \pi \int y^2 \, dx = \pi \int_e^{e^2} (2 - \ln x)^2 \, dx$

$= \pi \int_e^{e^2} \left(4 - 4\ln x + (\ln x)^2\right) dx$

$= \pi \left[ 4x - 4x\ln x + 4x + x(\ln x)^2 - 2x\ln x + 2x \right]_e^{e^2}$

$= \pi \left[ 10x - 6x\ln x + x(\ln x)^2 \right]_e^{e^2}$

$= \pi \left[ 10e^2 - 6e^2 \ln e^2 + e^2 (\ln e^2)^2 - 10e + 6e\ln e - e(\ln e)^2 \right]$

$= \pi \left[ 2e^2 - 5e \right] = \boxed{\pi e \left[ 2e - 5 \right]}$